5.6 Law of Cosines

Two cases remain in the list of conditions needed to solve an oblique triangle – SSS and SAS. When you are given 3 sides (SSS), or 2 sides and their included angle (SAS), none of the ratios in the Law of Sines would be complete. In such cases, you can use Law of Cosines.

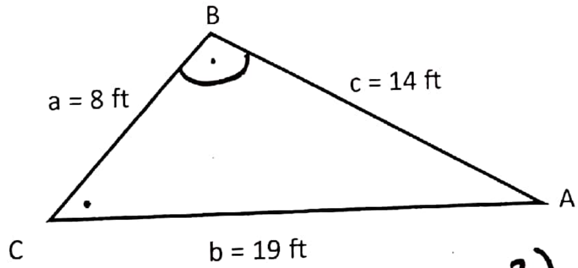
Law of Cosines

SAS	SSS
$a^2 = b^2 + c^2 - 2bc\cos A$	$\cos A = \dfrac{(b^2+c^2-a^2)}{(2bc)}$
$b^2 = a^2 + c^2 - 2ac\cos B$	$\cos B = \dfrac{(a^2+c^2-b^2)}{(2ac)}$
$c^2 = a^2 + b^2 - 2ab\cos C$	$\cos C = \dfrac{(a^2+b^2-c^2)}{(2ab)}$

Three Sides of a Triangle – SSS

Example 1: Find the 3 angles of the triangle shown below.

SSS



$$\cos B = \frac{(8^2 + 14^2 - 19^2)}{(2(8)(14))}$$

$(2 \times 8 \times 14)$

$\cos B = (-0.45\ldots)$

$B = \cos^{-1}(-0.45\ldots)$

$\boxed{B = 116.80^\circ}$

$$\cos C = \frac{(8^2 + 19^2 - 14^2)}{(2 \cdot 8 \cdot 19)}$$

$\cos C = 0.75\ldots$

$C = \cos^{-1}(0.75\ldots) = \boxed{C = 41.12^\circ}$

$A = 180 - (116.8 + 41.12)$

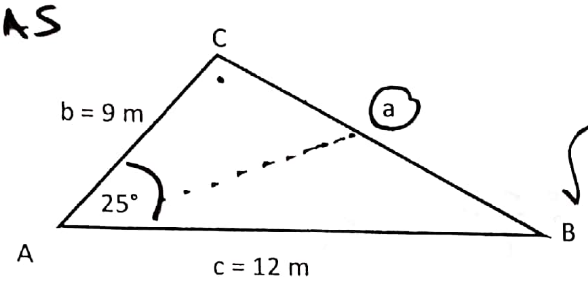
$\boxed{A = 22.08^\circ}$

You try: Find the 3 angles of the triangle whose sides have length a = 6, b = 8, and c = 12.

Two Sides and the Included Angle – SAS

Example 2: Use the Law of Cosines to find the remaining sides and angles of the triangle.

SAS



$$a^2 = 9^2 + 12^2 - 2\cdot 9\cdot 12\cos 25°$$

$$\sqrt{a^2} = \sqrt{29.23\ldots}$$

$$\boxed{a = 5.41\text{ m}}$$

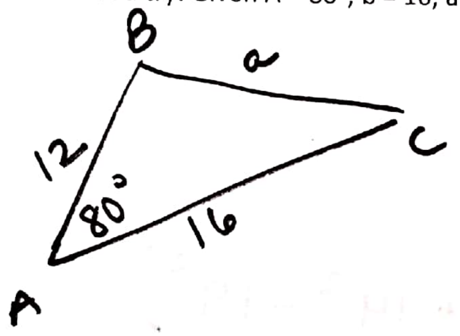
$$\cos C = \frac{9^2 + (5.407\ldots)^2 - 12^2}{2\cdot(5.407\ldots)(9)}$$

$$C = \cos^{-1}(-0.3468\ldots) = \boxed{110.30° = C}$$

$$\angle B = 180 - (25 + 110.297\ldots)$$

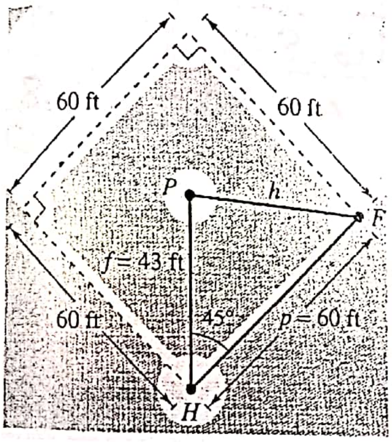
$$B = 44.70°$$

You try: Given A = 80°, b = 16, and c= 12, find the remaining angles and sides of the triangle.



Applications

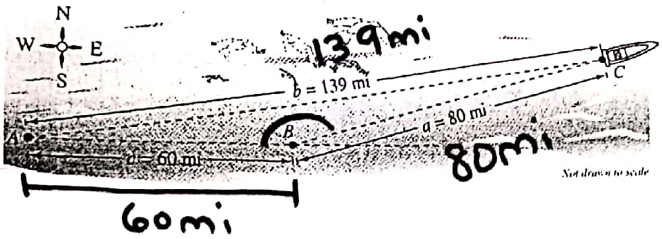
Example 3: The pitcher's mound on a women's softball field is 43 feet from home plate and the distance between the bases is 60 feet, as shown in the image below. (The pitcher's mound is *not* halfway between the home plate and second base.) How far is the pitcher's mound from first base?



Bearing N ____ E

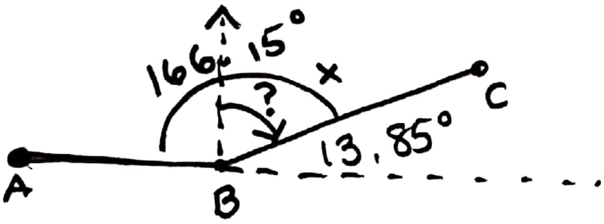
Example 4: A ship travels 60 miles due east and then adjusts its course, as shown below. After traveling for 80 miles in this new direction, the ship is 139 miles from its point of departure. Describe the bearing from point B to point C.

SSS



$$\cos B = \frac{(80^2 + 60^2 - 139^2)}{(2 \times 80 \times 60)}$$

$$B = 166.15°$$

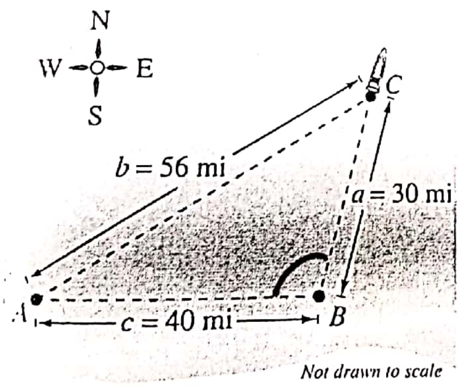


$$x + 13.85 = 90$$

$$x = 76.15°$$

Bearing from B to C $\underline{N\ 76.15°\ E}$

You try: A ship travels 40 miles due east and then changes direction, as shown in the figure below. After traveling 30 miles in this new direction, the ship is 56 miles from its point of departure. Describe the bearing from point B to Point C.



$$\cos B = \frac{(40^2 + 30^2 - 56^2)}{(2 \times 40 \times 30)}$$

The Law of Cosines can be used to establish the following formula for the area of a triangle. This formula is called **Heron's Area Formula** after the Greek mathematician Heron (ca. 100 B.C.)

Heron's Area Formula

SSS

Given any triangle with the side length a, b, and c, the area of the triangle is:

$$Area = \sqrt{s(s-a)(s-b)(s-c)}$$

where

$$* \; s = \frac{a+b+c}{2}$$

Example 5: Find the area of a triangle having sides of length a = 43 meters, b = 53 meters, and c = 72 meters.

$$S = \frac{(43+53+72)}{2}$$

$$S = 84$$

$$A = \sqrt{84\,(84-43)(84-53)(84-72)}$$

$$= \sqrt{(84)(41)(31)(12)}$$

$$= \boxed{1131.89\,m^2}$$

You try: Given a = 5, b = 9, and c = 8, use Heron's Area Formula to find the area of the triangle.